SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 9

SOLUTIONS

Problem 1. Show that if M_i are compact, connected, oriented manifolds of the same dimension for $i = 1, 2, 3$, and $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are C^∞ functions, then $\deg(g \circ f) = \deg(g)\deg(f)$.

Solution 1. Notice that the set of regular values of g and $g \circ f$ are both full measure subsets of M_3. Therefore, we may choose a point $y \in M_3$ such that y is a regular value of both $g \circ f$ and g. We claim that any $x \in g^{-1}(y)$ is also a regular value of f. Indeed, if $p \in f^{-1}(x)$, then $p \in (g \circ f)^{-1}(y)$. Since $d(g \circ f)(p) = dg(x) \circ df(p)$, and both $d(g \circ f)(p)$ and $dg(x)$ are invertible linear transfomrations (by the regularity assumption for y with respect to g and $g \circ f$, respectively), it follows that $df(p)$ must be invertible. Then

$$\begin{aligned}
\deg(g \circ f) &= \sum_{p \in (g \circ f)^{-1}(y)} \sigma(d(g \circ f)(p)) \\
&= \sum_{p \in (g \circ f)^{-1}(y)} \sigma(dg(f(p))\sigma(df(p)) \\
&= \sum_{x \in g^{-1}(y)} \sum_{p \in f^{-1}(x)} \sigma(dg(x))\sigma(df(p)) \\
&= \sum_{x \in g^{-1}(y)} \sigma(dg(x)) \sum_{p \in f^{-1}(x)} \sigma(df(p)) \\
&= \sum_{x \in g^{-1}(y)} \sigma(dg(x)) \deg(f) = \deg(g)\deg(f)
\end{aligned}$$

□

Solution 2. Fix a top form ω on M_3 such that $\int_{M_3} \omega = 1$. Then if $g^*\omega$ is a nonexact form on M_2:

$$\deg(g \circ f) = \int_{M_1} (g \circ f)^*\omega = \int_{M_1} f^*g^*\omega = \deg(f) \int_{M_2} g^*\omega = \deg(f)\deg(g) \int_{M_3} \omega = \deg(f)\deg(g)$$

On the other hand, if $g^*\omega$ is exact on M_2, then $\deg(g) = 0$ and $f^*g^*\omega$ is also exact. It follows that both sides of the equality are zero, and it holds in general. □

Problem 2. Show that if M and N are compact, connected and oriented manifolds of the same dimension, and $F : M \to N$ has $\deg(F) = 0$, then F has a critical point.

Solution. We prove the contrapositive. Assume that F has no critical point. Then $dF(x)$ is always non-degenerate, and $\sigma(dF(x))$ is locally constant for $x \in M$. Since M is connected, $\sigma(dF(x))$ is either always 1 or always -1. Hence any $y \in F(M) \subset N$ is a regular value and

$$\deg(F) = \sum_{x \in F^{-1}(y)} \sigma(dF(x)) = \pm\#F^{-1}(y)$$

Hence, $\deg(F) \neq 0$. □

Problem 3. Let A be an $n \times n$ square matrix with integer entries, $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ and $F_A : \mathbb{T}^n \to \mathbb{T}^n$ be defined by

$$F_A([x]) = [Ax]$$

where the notation $[x]$ denotes the equivalence class $x + \mathbb{Z}^n$.

(1) Show that F_A is well-defined.
(2) Compute $\deg(F_A)$ by computing the signed number of preimages of a regular value of F_A.
(3) Compute $\deg(F_A)$ by computing $\int F_A^*\omega$, where ω is the standard n-form $dx_1 \wedge \cdots \wedge dx_n$.

Solution.

(1) Assume that $x, y \in [x]$, so $y = x + q$ for $q \in \mathbb{Z}^n$. Then $Ay = A(x+q) = Ax + Aq$. Since $q \in \mathbb{Z}^n$ and A has integer entries, $Aq \in \mathbb{Z}^n$. Hence $[Ay] = [Ax]$, and A is well-defined.

(2) We claim that $\deg(F_A) = \det(A)$. First, we handle the case when $\det(A) = 0$. In this case, A has a kernel, so the image of F_A is an immersed manifold of smaller dimension. In particular, F_A cannot be onto, and there exists a regular value with no preimage. Hence $\deg(F_A) = 0$ as well.

Otherwise, $A : \mathbb{R}^n \to \mathbb{R}^n$ is onto, so F_A is onto. Furthermore, identifying all tangent spaces with $\mathbb{R}^n$. we see that $dF_A(x) = A$ for all $x \in \mathbb{T}^n$, so every value is regular and the sign is either always positive or always negative. It therefore suffices to show that the number of preimages of 0 is $|\det(A)|$.

Observe that $F_A(x) = 0$ if and only if $Ax \in \mathbb{Z}^n$, which is to say that $x \in A^{-1}\mathbb{Z}^n$. The number of preimages modulo $\mathbb{Z}^n$ is hence the index of $\mathbb{Z}^n$ inside $A^{-1}\mathbb{Z}^n$. That is, the index of $A\mathbb{Z}^n$ inside $\mathbb{Z}^n$. We claim that this is $|\det(A)|$. First, note that if $\psi(A)$ denotes the index of $A\mathbb{Z}^n$ inside $\mathbb{Z}^n$, then $\psi(AB) = \psi(A)\psi(B)$ and $\psi(\mathrm{Id}) = 1$. Furthermore, we know that $\psi(\mathrm{diag}(k_1, \dots, k_n)) = k_1 \dots k_n$ by direct calculation (the same is true for an upper triangular matrix). Indeed, one may immediately see that the quotient group here is isomorphic to $\mathbb{Z}/(k_1\mathbb{Z}) \times \cdots \times \mathbb{Z}/(k_n\mathbb{Z})$. Furthermore, if $E = E(i_0, j_0)$ is an elementary matrix, which has entries $E_{ij} = 1$ if and only if $i = j$ or $i = i_0$ and $j = j_0$, $\psi(E) = 1$. Finally, all permuatation matrices clearly have $\psi = 1$. Since every matrix is a product of upper triangular matrices, permutation matrices, and elementary matrices (Gaussian elimination for $\mathbb{Z}^n$ matrices), and ψ coincides with the (absolute value of the) determinant on such matrices, $\psi = |\det|$.

Remark 1. In class, I mentioned that one may use Cramer's rule and adjugate matrices to prove this. When I wrote this problem, this was the solution I had in mind, but I had forgotten it only works for $\deg = \pm 1$. I apologize for any confusion!

(3) Notice that $\int_{\mathbb{T}^n} = 1$, since a fundamental domain of the action of $\mathbb{Z}^n$ is $[0,1]^n$, which has n-dimensional volume 1. Furthermore,

$$F_A^*\omega(x)(e_1, \dots, e_n) = \omega(F_A(x))(dF_A(e_1), \dots, dF_A(e_n)) = \det(Ae_1, \dots, Ae_n) = \det(A)$$

Hence $F_A^*\omega = \det(A)\omega$, and it follows that $\int_{\mathbb{T}^n} F_A^*\omega = \det(A)$.

□